Working with Henry Wynn: Over 30 years of collaboration in a few slides (and photos)

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June 2026

1 Introduction

A short extract from our contribution to the paper
"*Working with Henry Wynn: a challenge and an exceptional opportunity*",
with Alessandra Giovagnoli, Hugo Maruri-Aguilar and Eva Riccomagno

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2 parts (not mentioning several European projects on "concurrent engineering")

1. Dynamical systems in search and optimisation ($\approx$ 1991–2008, $\rightarrow$ mODa 3 & 4)
2. Simplicial measures of multivariate dispersion ($\approx$ 2015–2024, $\rightarrow$ mODa 13)

First meetings (early 90's) in London, then in Antibes...

A pause between two (intensive) working sessions:

2 Dynamical systems in search and optimisation

Objective: search for a target $\mathbf{x}^*$ in a given region R_0

[for instance, $\mathbf{x}^* =$ the minimiser of a function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$]

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Consistency

→ "Observations" (evaluations of f) force restrictions on possible location of $\mathbf{x}^*$:

After n "observations" $y_1, \ldots, y_n$, we know that $\mathbf{x}^*$ belongs to a *consistency region* $R_n = \mathscr{X}(y_1, \ldots, y_n)$ (may have a complicated form)

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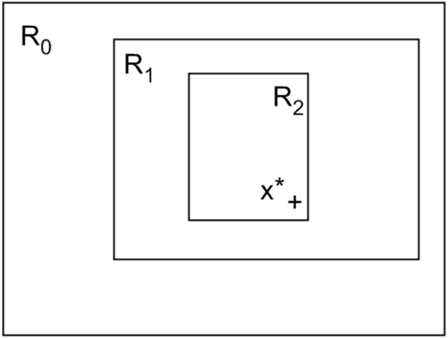
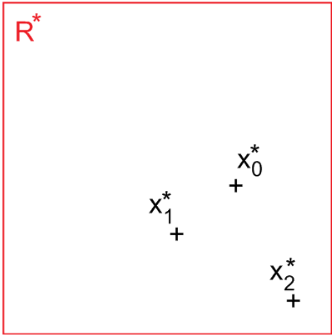
Renormalisation

Renormalise R_n (or a simpler outer approximation of it)

to a base region R^*

consistency ➠ renormalisation





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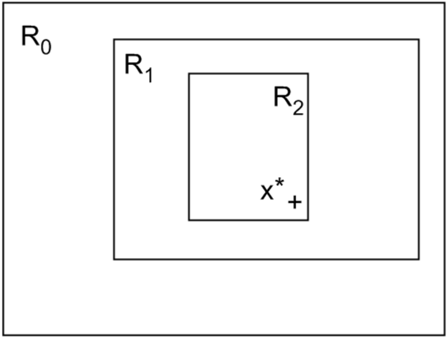

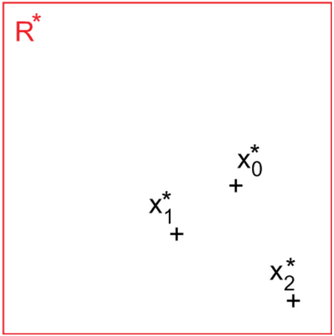

$\mathbf{x}^*$ fixed in R_0 ➠ $\boxed{\text{dynamical system } \mathbf{x}_0^* \to \mathbf{x}_1^* \to \mathbf{x}_2^* \to \cdots \text{ in } R^* \text{ fixed}}$

Consequences

Rate of convergence = rate of contraction of the R_i
= rate of expansion of the dynamical system
(Lyapunov exponents, entropies)

Ergodic behaviour, similar for almost any initialisation of the algorithm
→ ergodically, worst-cases events have measure zero
(we can improve upon algorithms considered as worst-case optimal)

The book (P, W & Z, 2000)* tells all the story...

* P, W & Z, 2000. *Dynamical Search*. Chapman & Hall/CRC, Boca Raton

$d = 1$: Second-order line search

Objective: minimise f having a unique minimiser x^* in an interval $[A_0, B_0)$

$\rightarrow$ compare $f(\cdot)$ at U_n and V_n in $[A_n, B_n)$, $U_n < V_n$

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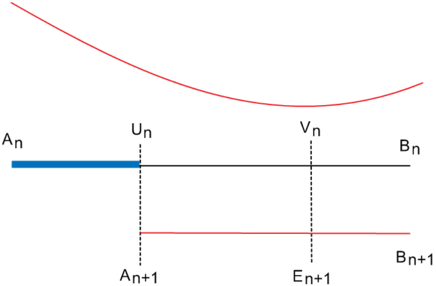
If $f(U_n) \geq f(V_n)$, delete $[A_n, U_n)$ (Left deletion)



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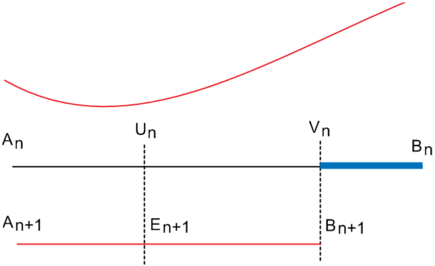

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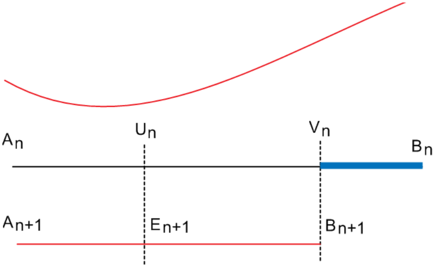

Next interval $[A_{n+1}, B_{n+1})$ contains U_n or V_n,

➞ let E_{n+1} denote this point; choose a new point E'_{n+1} in $[A_{n+1}, B_{n+1})$; repeat...

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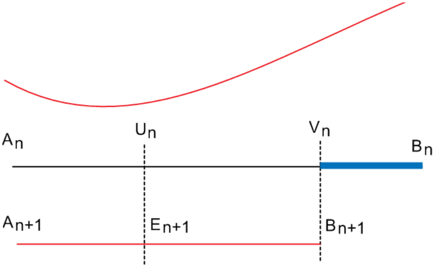

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Renormalise $[A_n, B_n)$ to $R^* = [0, 1)$: $A_n \to 0$, $B_n \to 1$, $E_n \to e_n$, $E'_n \to e'_n$

Golden Section algorithm (Kiefer, 1957)*:

$$x^*_{n+1} = \begin{cases} (1+\phi)x^*_n & \text{if } x^*_n < 1/2 \\ (1+\phi)x^*_n - \phi & \text{if } x^*_n \geq 1/2 \end{cases}$$

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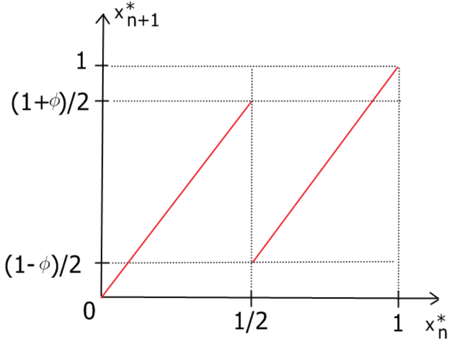
GS iteration



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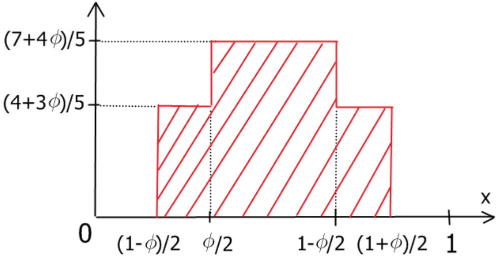
invariant density for the GS algorithm



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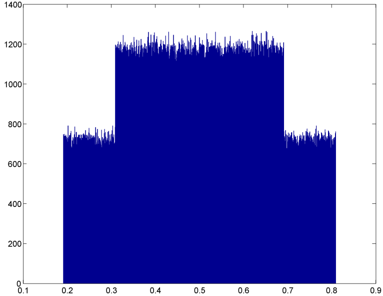
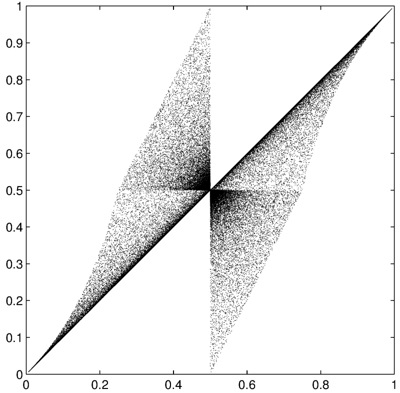


Other second-order line-search algorithms?

Symmetric algorithms: $\boxed{e'_n = 1 - e_n}$ for all n (with GS as a particular case)

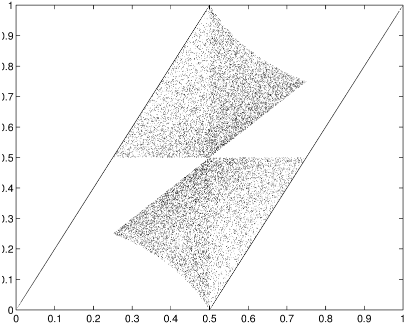
(x_n^*, x_{n+1}^*), $n = 1, \ldots, 100\,000$, for a typical symmetric algorithm



Midpoint algorithm: $\boxed{e'_n = 1/2}$ for all n

(assume $f(\cdot)$ locally symmetric at x^*)

(x^*_n, x^*_{n+1}), $n = 1, \ldots, 50\,000$

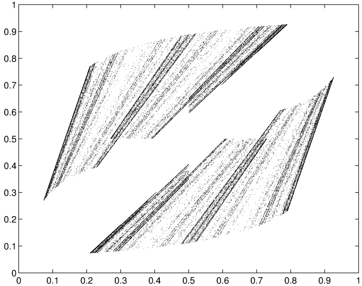


(faster than GS)

Window algorithm: $\boxed{e_n' = e_n + w \text{ if } e_n < \frac{1}{2} \text{ and } e_n' = e_n - w \text{ if } e_n \geq \frac{1}{2}}$

(assume $f(\cdot)$ locally symmetric at x^*)

(x_n^*, x_{n+1}^*), $n = 1, \ldots, 50\,000$ ($w = 1/8$)

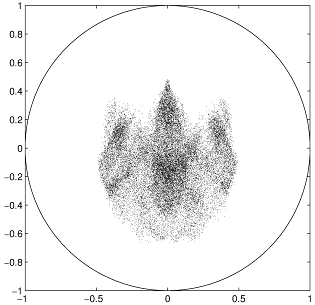


(the fastest we found)

$d > 1$: Ellipsoid algorithm?

Khachiyan (1979)* algorithm for linear programming:

Typical sequence $\mathbf{x}_n^*$



→ too complicated!

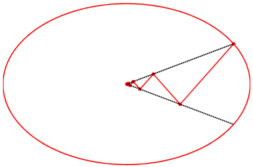
* Khachiyan, L., 1979. A polynomial algorithm in linear programming. *Doklady AkademïaNauk SSSR* **244**, 1093–1096, translated into English in *Soviet Mathematics Doklady*, **20**, 191–194

$d > 1$: Steepest descent to minimise a quadratic function!

SD: $\boxed{\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha_k \nabla_f(\mathbf{x}^{(k)})}$, with $\alpha_k = \arg\min_\alpha f[\mathbf{x}^{(k)} - \alpha \nabla_f(\mathbf{x}^{(k)})]$

Quadratic function: $f(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^\top \mathbf{A}(\mathbf{x} - \mathbf{x}^*)$, $\mathbf{A}$ pos. def.

Known result in $\mathbb{R}^d$ (Akaike, 1959)*
(Akaike's bird cage)



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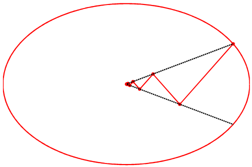

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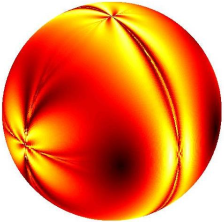
...but detailed behaviour is rather complicated even in finite (and small) dimension

$d = 3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$
the colour (of starting point on $\mathbb{S}_2(0,1)$) depends on limiting p



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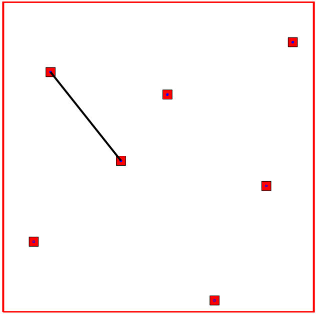
Another pause between two (intensive) working sessions:

3 Simplicial measures of multivariate dispersion

Take n points x_i i.i.d. (prob. measure μ) in $\mathbb{R}^d$
Compute squared distances between all pairs :

$$\frac{2}{n(n-1)} \sum_{i<j} \|x_i - x_j\|^2 = 2\,\mathrm{trace}[\widehat{\mathbf{V}}_n]$$

with $\widehat{\mathbf{V}}_n$ the empirical covariance matrix



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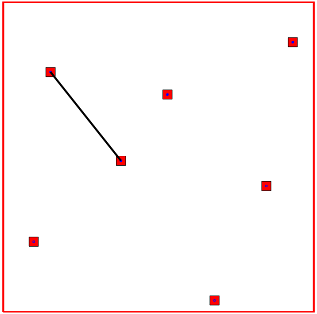
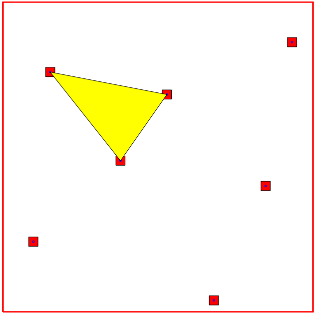


$$\int\int \|x_1 - x_2\|^2 \,\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2) = 2\,\mathrm{trace}(\mathbf{V}_\mu)$$

Take n points x_i i.i.d. (prob. measure μ) in $\mathbb{R}^d$

Compute squared area of triangles for all triples :

$$\frac{6}{n(n-1)(n-2)} \sum_{i<j<k} \mathscr{V}_2^2(x_i, x_j, x_k) = ?$$



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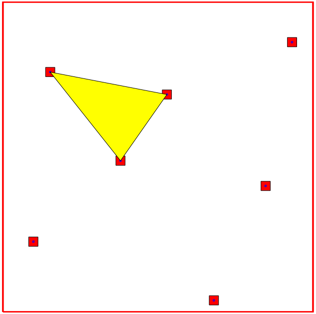

$$\psi_2(\mu) = \int\int\int \mathscr{V}_2^2(x_1, x_2, x_3)\, \mu(\mathrm{d}x_1)\, \mu(\mathrm{d}x_2)\, \mu(\mathrm{d}x_3) = ?$$

What about volumes of k-dimensional simplices in $\mathbb{R}^d$, $k \leq d$?

Which properties in terms of functionals of μ?

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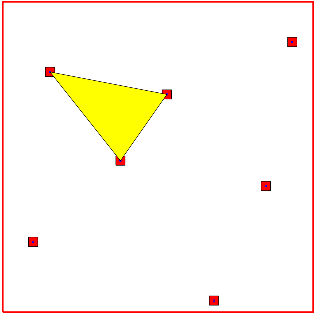

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What about using other powers δ:

$$\psi_{k,\delta}(\mu) = \int \cdots \int \mathscr{V}_k^\delta(x_1, \ldots, x_{k+1})\, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_{k+1}) \ ?$$

$\delta = 2$: $\mu \in \mathscr{M} \longrightarrow \psi_{k,2}(\mu) = \mathsf{E}_\mu\{\mathscr{V}_k^2(x_1, \ldots, x_{k+1})\}$

$k = 2$ ➠ triangles $\psi_2(\mu) = \int\int\int \mathscr{V}_2^2(x_1, x_2, x_3)\, \mu(\mathrm{d}x_1)\, \mu(\mathrm{d}x_2)\, \mu(\mathrm{d}x_3)$

$$\psi_{k,2}(\mu) = \Psi_{k,2}[\mathbf{V}_\mu] = \frac{k+1}{k!} \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}[\mathbf{V}_\mu] \times \cdots \times \lambda_{i_k}[\mathbf{V}_\mu]\,,$$

$\psi_{k,2}^{1/k}(\cdot)$ is shift-invariant, homogeneous of degree 2 and concave
→ optimal design, equivalence theorem, etc.

(López-Fidalgo and Rodríguez-Díaz, 1998)[1]
(P, W & Z, 2017)[2], (P, W & Z, mODa 12)

[1] López-Fidalgo, J., Rodríguez-Díaz, J., 1998. Characteristic polynomial criteria in optimal experimental design. In *Proc. of MODA'5*, Marseilles, June 22–26, 1998. Physica Verlag, Heidelberg, pp. 31–38

[2] P, W & Z, 2017. Extended generalised variances, with applications. *Bernoulli* **23** (4A), 2617–2642, arXiv:1411.6428, hal-01308092

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$\delta \neq 2$: $\mu \in \mathscr{M} \longrightarrow \psi_{k,\delta}(\mu) = \mathsf{E}_\mu\{\mathscr{V}_k^\delta(x_1,\ldots,x_{k+1})\}$

- $\delta > 0$ (maximise $\psi_{k,\delta}$): $k = 1$ in (Björck, 1956)[3], $k \geq 2$ ➠ some results. . .
- $\delta < 0$ (minimise $\psi_{k,\delta}$): $k = 1 \leftrightarrow$ potential theory, $k \geq 2$ largely open. . .

[3] Björck, G., 1956. Distributions of positive mass, which maximize a certain generalized energy integral. *Arkiv för Matematik* **3** (21), 255–269

Example: $\delta = -\infty$ ➞ maximise $\min_{\{x_1,\ldots,x_{k+1}\}\subset\mathbf{X}_n} \mathscr{V}_k(x_1,\ldots,x_{k+1})$

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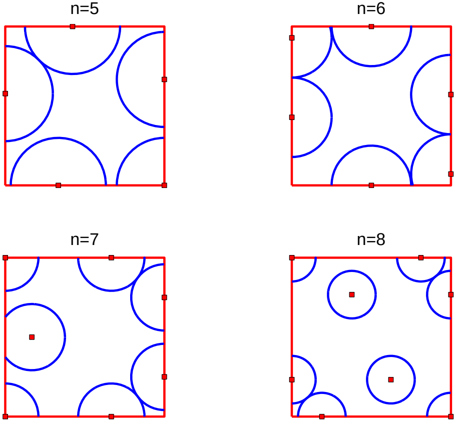
$$k = d = 2:$$



(P, W & Z ProbaStat'2015)

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Let's celebrate all these years: cheers, Henry!